Planar Multiple-Valued Decision Diagrams

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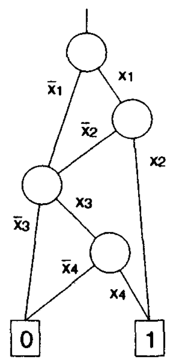

Figure 1.1: An r-planar BDD.

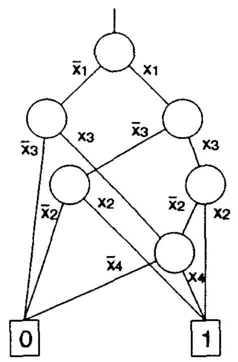

Figure 1.2: A non r-planar BDD.

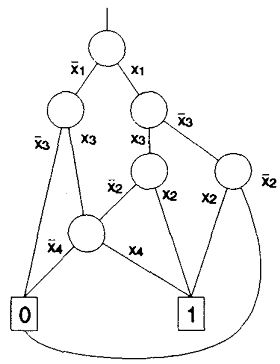

Figure 1.3: A planar drawing of BDD.

Abstract

In VLSI, crossings occupy space and cause delay. Therefore, there is significant benefit to planar circuits. We propose the use of planar multiple-valued decision diagrams to produce planar multiple-valued circuits. Specifically, we show conditions on 1) threshold functions, 2) symmetric functions, and 3) monotone increasing functions that produce planar decision diagrams. Our results apply to binary functions, as well. For example, we show that all two-valued monotone increasing threshold functions of up to five variables have planar binary decision diagrams.

Index terms: binary decision diagram (BDD), dual function, threshold function, field programmable gate array (FPGA).

1 Introduction

Multiple-valued decision diagrams (MDDs) are multiple-valued extensions of binary decision diagrams (BDDs). MDDs are useful for designing multiple-valued logic networks; by replacing each node of an MDD with a multiple-valued multiplexer (MUX), we have a multiple-valued network for the function.

Fig. 1.1 shows a BDD for $f = x_1x_2 \vee x_3x_4$. This BDD has no crossing, which we denote as **r-planar.** Fig. 1.2 shows a BDD for the same function with a different ordering of the input variables. In this case, however, the BDD is not r-planar, since it has crossings. We say a function has an r-planar BDD if we can draw a planar BDD in a restricted form:

Definition 1.1 *A BDD in which*

1. *a 1-edge emerges to the right of the node,*
2. *a 0-edge emerges to the left, and*
3. *the constant 1 node is to the left of the constant 0 node*

is **r-planar** *(restricted-planar) if it has no crossings.*

r-planar graphs are special case of planar graphs. Fig. 1.3 shows a planar BDD that is isomorphic to the BDD in Fig. 1.2, which is not an r-planar BDD. Fig. 1.4 shows a network for $f = x_1x_2 \vee x_3x_4$. It corresponds to the BDD in Fig. 1.1, where each node in the BDD is replaced with a binary MUX. Note that this network has no crossings if we ignore the lines for the input variables. Fig. 1.5 is a network that corresponds to the BDD in Fig. 1.2. In this case, the network has crossings. When we implement networks in the form of LSIs, crossings are often expensive; they require additional channels and increase delay. Especially in the case of field programmable gate arrays (FPGAs) [2], crossings produce considerable delay. Since the delay of interconnections is the most important problem in

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14. ABSTRACT
In VLSI, crossings occupy space and cause delay. Therefore, there is significant benefit to planar circuits. We propose the use of planar multiple-valued decision diagrams to produce planar multiple-valued circuits. Specifically, we show conditions on 1) threshold functions, 2 symmetric functions, and 3) monodiagrams. Our results apply to binary functions, as well. For example, we show that all two-valued monotone increasing threshold functions of up to five variables have planar binary decision diagrams.

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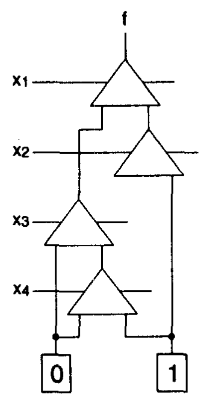


Figure 1.4: An MUX network corresponding to Fig. 1.1.

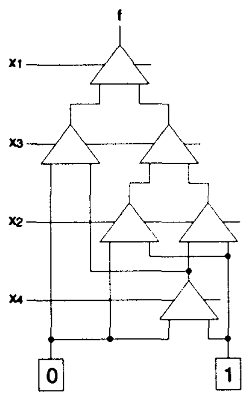


Figure 1.5: An MUX network corresponding to Fig. 1.2.

FPGA design, networks without crossing are quite attractive. Also, in sub-micron LSIs, networks without crossings are desirable, since the delays in the interconnections and crossing are comparable to the delay for logic elements.

In this paper, we identify classes of logic functions whose MDDs and BDDs are r-planar. For these functions, we can easily derive logic networks whose layouts are relatively simple. Initially, we consider unrestricted MDDs and BDDs. Subsequently, we consider reduced ordered MDDs and BDDs that do not contain redundant nodes nor nodes representing the same function.

2 r-Planar MDDs

In this section, we define multiple-valued input two-valued output functions [11]. Then, we show some classes of functions having r-planar MDDs. These results will be used for the identification of functions having r-planar BDDs in Section 3. As for the definitions for BDDs and MDDs, refer to [1, 3, 7].

Definition 2.1 *A* **multiple-valued input two-valued output function** *is*

$$g(x_1, x_2, \ldots, x_n) : \mathop{\times}_{i=1}^{n} P_i \rightarrow B,$$

where x_i assumes a value in $P_i = \{0, 1, \ldots, p_i - 1\}$ and $B = \{0, 1\}$.

Definition 2.2 *Let x_i be a variable taking values in $P_i = \{0, 1, \ldots, p_i - 1\}$. Let S_i be a subset of P_i. Then, $x_i^{S_i}$ is a* **literal** *of S_i, where $x_i^{S_i} = 1$ if $x_i \in S_i$, and $x_i^{S_i} = 0$ otherwise. When S_i contains only one element $a \in P_i$, $x_i^{\{a\}}$ is written as x_i^a.*

Lemma 2.1 *A multiple-valued input two-valued output function f can be represented by an expression*

$$f(x_1, x_2, \ldots, x_n) = \bigvee_{(S_1, S_2, \ldots, S_n)} x_1^{S_1} x_2^{S_2} \cdots x_n^{S_n},$$

where $\vee$ is OR and concatenation is AND.

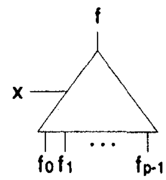


Figure 2.1: Multiple-valued MUX.

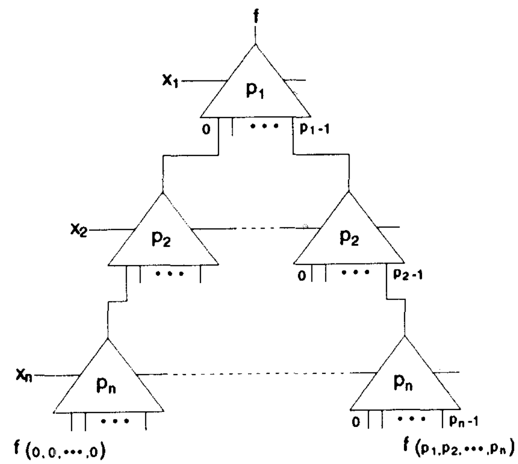


Figure 2.2: Tree network with multiple-valued MUXs.

A multiple-valued multiplexer (MUX), shown in Fig. 2.1, selects one terminal according to the value of x, where $x \in \{0, 1, \ldots, p-1\}$. The function of the MUX is represented by

$$f(x) = x^0 f_0 \vee x^1 f_1 \vee \cdots \vee x^{p-1} f_{p-1}.$$

Lemma 2.2 *The tree network of MUXs shown in Fig. 2.2 realizes an arbitrary multiple-valued input two-valued output function.*

Definition 2.3 *Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be vectors such that $a_i, b_i \in \{0, 1, \ldots, p_i - 1\}$. We define a binary relation $\preceq$ between vectors as follows: $\boldsymbol{a} \preceq \boldsymbol{b}$ iff $\boldsymbol{a}$ appears before $\boldsymbol{b}$ in lexicographical order.*

For example, $(0,0,0) \preceq (0,0,1)$, and $(0,1,1) \preceq (1,0,0)$.

Definition 2.4 *A function $f(x)$ is* **l-monotonic** *(lexicographically monotonic) iff the following conditions hold: For vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$, such that $a_i, b_i \in \{0, 1, \ldots, p_i - 1\}$, $\boldsymbol{a} \preceq \boldsymbol{b}$, implies $f(\boldsymbol{a}) \leq f(\boldsymbol{b})$, where the logic values are viewed as integers. $f(X) \subseteq g(X)$ iff $f(\boldsymbol{a}) \leq g(\boldsymbol{a})$ for any $\boldsymbol{a}$.*

Lemma 2.3 *Suppose that a function f is l-monotonic. Let $X_1 = (x_1, x_2, \ldots, x_i)$, and $X_2 = (x_{i+1}, x_{i+2}, \ldots, x_n)$ be a partition of $X = (x_1, x_2, \ldots, x_n)$. Then, $f(\boldsymbol{a}, X_2) \subseteq f(\boldsymbol{b}, X_2)$ for any $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ such that $\boldsymbol{a} \preceq \boldsymbol{b}$.*

(Proof) Suppose that for some $\boldsymbol{c} = (c_{i+1}, c_{i+2}, \ldots, c_n)$, $f(\boldsymbol{a}, \boldsymbol{c}) > f(\boldsymbol{b}, \boldsymbol{c})$ holds. Because $\boldsymbol{a} \preceq \boldsymbol{b}$, we have $(\boldsymbol{a}, \boldsymbol{c}) \preceq (\boldsymbol{b}, \boldsymbol{c})$. However, this contradicts the assumption that f is l-monotonic. Thus, there are no vector $\boldsymbol{c}$ that satisfies $f(\boldsymbol{a}, \boldsymbol{c}) > f(\boldsymbol{b}, \boldsymbol{c})$. (Q.E.D.)

Definition 2.5 *A complete MDD is an MDD that has a distinct node for every assignment of values to the variables. That is, no two nodes are merged.*

Definition 2.6 *Let f be a p-valued input two-valued output function. An MDD for f in which*

1. *an i-edge emerges to the right of an $(i-1)$-edge, $(1 \leq i \leq p-1)$, and*
2. *the constant 1 node is to the left of the constant 0 node*

is r-planar (restricted-planar) if it has no crossings.

Lemma 2.4 *An l-monotonic function has an r-planar complete MDD.*

Definition 2.7 *A* **reduced ordered multiple-valued decision diagram** *(ROMDD) is an MDD where*

1. *two nodes are merged into one node if they represent the same function, and*
2. *a node η is removed if all the children of η represent the same function.*

Lemma 2.5 *An l-monotonic function has an r-planar ROMDD.*

(Proof) Consider a complete MDD of function f, as shown in Fig. 2.2. Because f is l-monotonic, by Lemma 2.3, if $\boldsymbol{a} \preceq \boldsymbol{b}$ then $f(\boldsymbol{a}, X_2) \subseteq f(\boldsymbol{b}, X_2)$. The functions represented by the nodes at the same level are totally ordered. In the lowest level, they are constant 0 or 1. From Lemma 2.4, the complete MDD for f is r-planar. Now, reduce the complete MDD into an ROMDD.

First, merge two nodes that represent same logic function. We show that the resulting MDD is also r-planar. Suppose that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, and $\boldsymbol{e}$ are nodes in the same level, where $\boldsymbol{a} \preceq \boldsymbol{b} \preceq \boldsymbol{c} \preceq \boldsymbol{d} \preceq \boldsymbol{e}$. Also, suppose that $\boldsymbol{b}$ and $\boldsymbol{d}$ have the property,

$$f(\boldsymbol{b}, X_2) = f(\boldsymbol{d}, X_2). \tag{2.1}$$

Fig. 2.3(a) shows the situation. Because f is l-monotonic, we have

$$f(\boldsymbol{b}, X_2) \subseteq f(\boldsymbol{c}, X_2) \subseteq f(\boldsymbol{d}, X_2). \tag{2.2}$$

From, (2.1) and (2.2), we have

$$f(\boldsymbol{b}, X_2) = f(\boldsymbol{c}, X_2) = f(\boldsymbol{d}, X_2).$$

This shows that the sub-tree for $\boldsymbol{c}$ also represents the same function as $\boldsymbol{b}$ and $\boldsymbol{d}$. Thus, these three sub-trees can be merged into one as shown in Fig. 2.3(b). Note that this operation does not introduce a crossing. It follows that merging two nodes that represent the same function preserves r-planarity. Also, it is clear that the reduction of redundant nodes preserves r-planarity. Hence, we have the lemma. (Q.E.D.)

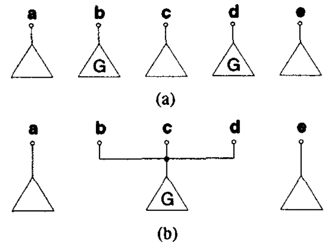


Figure 2.3: Derivation of r-planar MDD.

Definition 2.8 *A multiple-valued input two-valued output function f is a* **threshold function** *if f can be represented as*

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} w_i x_i \geq T, \\ 0 & \text{otherwise.} \end{cases}$$

where w_i is a **weight** *for the variable $x_i (i = 1, 2, \ldots, n)$, and T is the* **threshold** *of the function. The threshold function f is represented by the* **characteristic vector** $(w_1, w_2, \ldots, w_n : T)$.

Theorem 2.1 *Let f be a multiple-valued input two-valued output threshold function whose characteristic vector $(w_1, w_2, \ldots, w_n : T)$ satisfies $w_i \geq \sum_{k=i+1}^{n} w_k(p_k - 1)$, and $w_1 \geq 1$, Then, f has an r-planar ROMDD.*

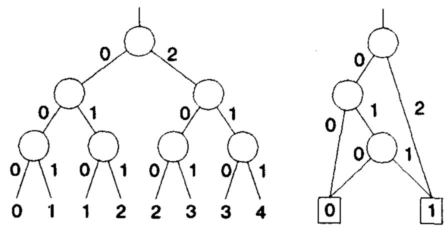
(Proof) Consider two vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$, such that $\boldsymbol{a} \preceq \boldsymbol{b}$. From the hypothesis of the theorem, we have

$$w_i x_i \geq \sum_{k=i+1}^{n} w_k x_k,$$

when $x_i \geq 1$. Since $\boldsymbol{a} \preceq \boldsymbol{b}$, we have $\sum_{i=1}^{n} a_i w_i \leq \sum_{i=1}^{n} b_i w_i$. Thus, $f(\boldsymbol{a}) \leq f(\boldsymbol{b})$, and f is l-monotone. By Lemma 2.5, f has an r-planar ROMDD. (Q.E.D.)

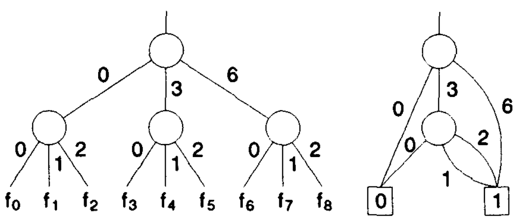
Example 2.1 *Consider the two-valued input threshold function $f(x_1, x_2)$ with the characteristic vector $(w_1, w_2, w_3 : T) = (2, 1, 1 : T)$. Note that this function satisfies the conditions of Theorem 2.1. Thus, f has an r-planar BDD. Note that f represents the functions $f = x_1 x_2 x_3$ when $T = 4$, $f = x_1(x_2 \vee x_3)$ when $T = 3$, $f = x_1 \vee x_2 x_3$ when $T = 2$, $f = x_1 \vee x_2 \vee x_3$ when $T = 1$, and $f = 1$ when $T = 0$. Fig. 2.4(a) is the complete decision tree with weighted edges. Fig. 2.4(b) is the ROBDD for $T = 2$.* *(End of Example)*

Example 2.2 *Consider the three-valued input threshold function $f(x_1, x_2, x_3)$ with the characteristic vector $(w_1, w_2 : T) = (3, 1 : T)$. Note that this function satisfies the conditions of Theorem 2.1. Thus, f has an r-planar MDD. Fig. 2.5(a) is the complete decision tree with weighted edges. Fig. 2.5(b) shows the ROMDD for $T = 4$.* *(End of Example)*



(a) Complete decision tree with weights. (b) ROBDD.

Figure 2.4: Derivation of r-planar ROBDD for threshold function.



(a) Complete decision tree. (b) ROMDD.

Figure 2.5: Derivation of r-planar ROMDD for threshold function.

Example 2.3 *Consider the two-valued input function:* $f = x_1 \vee x_2(x_3 \vee x_4)$. *Note that* f *is a threshold function with the characteristic vector* $(w_1, w_2, w_3, w_4 : T) = (5, 3, 1, 1 : 4)$. *This vector satisfies the condition of Theorem 2.1. So, the function with the ordering* (x_1, x_2, x_3, x_4) *has an* r*-planar ROBDD, as shown in Fig. 2.6(a). A different ordering* (x_4, x_1, x_3, x_2) *produces a non* r*-planar ROBDD, as shown in Fig. 2.6(b).* *(End of Example)*

Theorem 2.2 *Suppose that a multiple-valued input two-valued output function* f *can be represented as*

$$f = X^A \cdot g \text{ or } f = X^A \vee g,$$

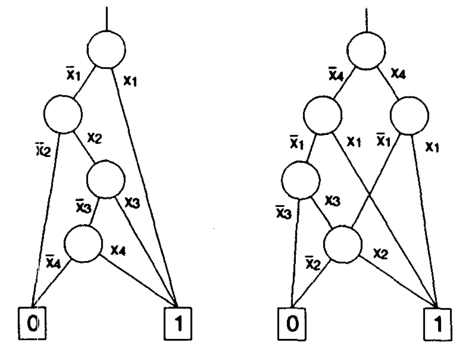
where X *takes a value in* $P = \{0, 1, \ldots, p-1\}$, $A = \{a, a+1, \ldots, p-1\}$, $(1 \leq a \leq p-1)$, *and* g *does not depend on* X. *If* g *has an* r*-planar MDD, then* f *has an* r*-planar MDD.*

(Proof) Fig. 2.7(a) and (b) show r-planar MDDs for $f = X^A \cdot g$ and $f = X^A \vee g$, respectively. (Q.E.D.)

3 r-planar BDD

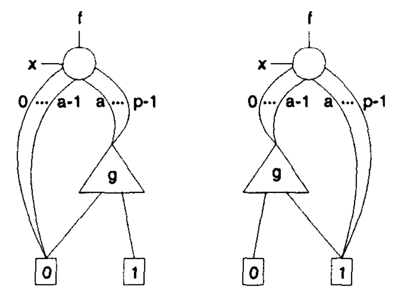
In this section, we consider the class of two-valued input two-valued output functions having r-planar ROBDDs. Here, for simplicity, *function* means two-valued input two-valued output function, unless otherwise noted.

Definition 3.1 *A* **complete symmetric decision diagram** *(Fig. 3.1) is the decision diagram on variables* $x_1, x_2, \ldots,$ *and* x_n *that has* $n+1$ *leaf nodes* $v_0, v_1, \ldots,$ *and* v_n, *such that* v_i *can be reached by only an assignment of values to* $X = (x_1, x_2, \ldots, x_n)$ *whose weight (number of 1's) is* i.



(a) r-planar ROBDD. (b) non r-planar ROBDD.

Figure 2.6: ROBDDs for $f = x_1 \vee x_2(x_3 \vee x_4)$.



(a) r-planar MDD for $f=X^A \cdot g$. (b) r-planar MDD for $f=X^A \vee g$.

Figure 2.7: Decomposition of r-planar MDD.

Fig. 3.2(a), (b) and (c) show the complete symmetric decision diagrams for $n = 1$, 2 and 3, respectively. Note that they are planar, and, in general, we have the following:

Lemma 3.1 *A complete symmetric decision diagram has an* r*-planar ROBDD.*

Definition 3.2 *A* **voting function** $S_k(X)$ *is a totally symmetric threshold function that can be represented as:*

$$S_k(X) = \begin{cases} 1 & \text{if } \|X\| \geq k \\ 0 & \text{otherwise,} \end{cases}$$

where $\|X\|$ *represents the weight (number of 1's) in the inputs* X.

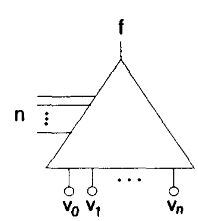
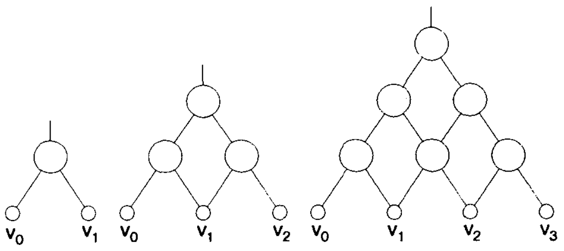
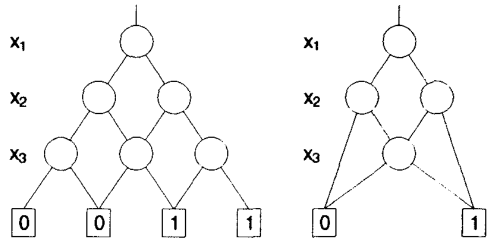


Figure 3.1: Complete symmetric decision diagram.



(a) n=1. (b) n=2. (c) n=3.

Figure 3.2: Complete symmetric decision diagrams for $n = 1, 2, 3$.



(a) Complete BDD. (b) ROBDD.

Figure 3.3: Derivation of ROBDD for a voting function.

Lemma 3.2 *A voting function has an r-planar ROBDD.*

(Proof) An ROBDD for an n variable voting function is derived from the complete symmetric decision diagram for n variables by assigning 0 to leaf nodes v_0 to v_i, and 1 to leaf nodes v_{i+1} through v_n. Reduction operations (e.g. merging v_0 through v_i, and v_{i+1} through v_n) preserves r-planarity. (Q.E.D.)

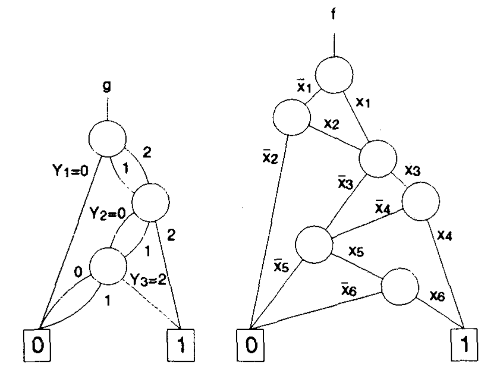
Example 3.1 *Fig. 3.3 shows the construction described in the proof for $n = 3$. (End of Example)*

Definition 3.3 *Let $X = (X_1, X_2, \ldots, X_r)$ be a partition of $X = (x_1, x_2, \ldots, x_n)$. A function f is* **partially symmetric** *with respect to $X_i (i = 1, 2, \ldots, r)$ if f is invariant under any permutation of the variables in X_i.*

Lemma 3.3 *Let f be a partially symmetric function with respect to X_i, where X_i contains n_i variables $(i = 1, 2, \ldots, r)$. Then, f is represented by a multiple-valued input two-valued output function $g(Y_1, Y_2, \ldots, Y_n)$, where Y_i takes one of $n_i + 1$ values representing the number of 1's in X_i.*

Definition 3.4 *The multiple-valued input two-valued output function g that corresponds to the partially symmetric function f in Lemma 3.3, is called a* **companion function** *of f.*

Theorem 3.1 *A partially symmetric function has an r-planar ROBDD if the companion function has an r-planar ROMDD.*



(a) r-planar MDD for $g = Y_1^{\{1,2\}}(Y_2^{\{2\}} \vee Y_3^{\{2\}})$. (b) r-planar BDD for $f = (x_1 \vee x_2)(x_3x_4 \vee x_5x_6)$.

Figure 3.4: Derivation of r-planar BDD.

(Proof) Suppose that the r-planar MDD for the companion function g is given. By replacing each node of the MDD with a complete symmetric decision diagram, we can make a BDD for the partially symmetric function f. By Lemma 3.1, the complete symmetric decision diagram is an r-planar BDD. Thus, the BDD for f is also r-planar. (Q.E.D.)

Example 3.2 *$f = (x_1 \vee x_2)(x_3x_4 \vee x_5x_6)$ is partially symmetric with respect to $X_1 = (x_1, x_2)$, $X_2 = (x_3, x_4)$ and X- (x_5, x_6). Let*

$$\begin{aligned} Y_i &= 0 \quad \text{if} \quad X_i = (0,0) \\ Y_i &= 1 \quad \text{if} \quad X_i = (0,1) \text{ or } X_i = (1,0), \text{ and} \\ Y_i &= 2 \quad \text{if} \quad X_i = (1,1). \end{aligned}$$

Then, the companion function g is represented by

$$g(Y_1, Y_2, Y_3) = Y_1^{\{1,2\}}(Y_2^{\{2\}} \vee Y_3^{\{2\}}). \tag{3.1}$$

By Theorem 2.2, we can see that g has an r-planar MDD. Fig. 3.4(a) shows the r-planar MDD for g. By replacing each node with an r-planar BDD, we have an r-planar BDD for f, as shown in Fig. 3.4(b). Note that f is not a threshold function. Also, note that companion functions can be generated iteratively. For example, (3.1) can be written as

$$h(Y_1, Z_1) = Y_1^{\{1,2\}} \cdot Z_1^{\{1,2\}},$$

where

$$Z^{\{1,2\}} = Y_2^{\{2\}} \vee Y_2^{\{2\}}.$$

In this way, companion functions can be constructed from other companion functions. (End of Example)

Lemma 3.4 *A function f has an r-planar ROBDD iff f^d has an r-planar ROBDD, where f^d is the dual function of f.*

(Proof) Suppose that f has an r-planar ROBDD. In the BDD, for each node, interchange the 0-edge and 1-edge. Also, interchange the constant 0 and the constant 1. Then, the resulting ROBDD represents f^d, and it is also r-planar. (Q.E.D.)

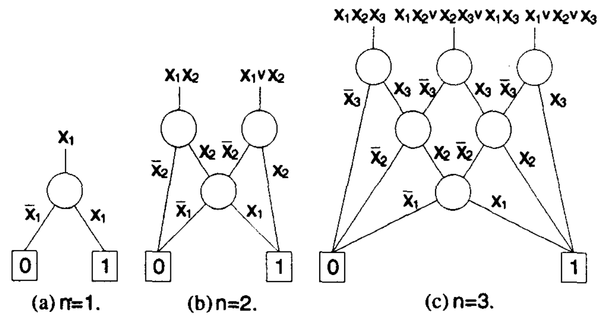


Figure 3.5: r-planar BDDs for voting functions.

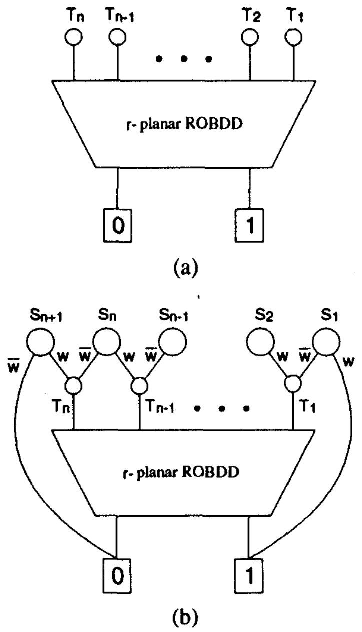


Figure 3.6: Generation of r-planar BDDs for voting functions.

Lemma 3.5 *Let $S_k(X)$ be a voting function such that*

$$S_k(X) = \begin{cases} 1 & \text{if } \|X\| \geq k \\ 0 & \text{otherwise.} \end{cases}$$

There exists an r-planar ROBDD that produces $S_0(X), S_1(X), \ldots,$ and $S_n(X)$, simultaneously.

(Proof) For $n = 1$, $n = 2$, and $n = 3$, the voting functions are generated as shown in Fig. 3.5(a), (b), and (c), respectively. Assume that Fig. 3.6(a) is an r-planar ROBDD that generates all the voting functions of n-variables. Then, we can make an r-planar ROBDD that generates all the voting functions of $(n+1)$-variables as shown in Fig. 3.6(b). (Q.E.D.)

Theorem 3.2 *Suppose that $X = (X_1, X_2)$ is a partition of variables $X = (x_1, x_2, \ldots, x_n)$, where $X_1 = (x_1, x_2, \ldots, x_k)$ and $X_2 = (x_{k+1}, x_{k+2}, \ldots, x_n)$. Let $\psi_i(X_1)$ be the symmetric function,*

$$\psi_i(X_1) = \begin{cases} 1 & \text{if } \|X_1\| = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $S_k(X_2)$ be a voting function. If a function f can be represented as

$$f(X_1, X_2) = \bigvee_{i=0}^{k} \psi_i(X_1) S_{a_i}(X_2), \qquad (3.2)$$

where $S_{a_i}(X_2) \subseteq S_{a_{i+1}}(X_2)$, then f has an r-planar ROBDD.

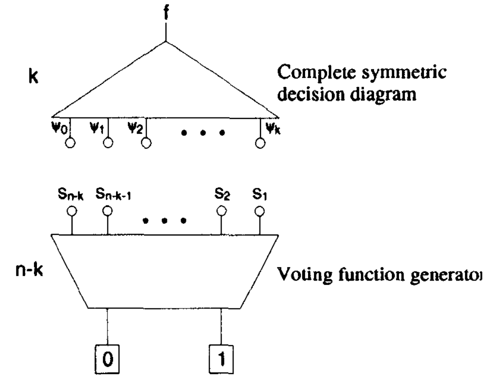


Figure 3.7: Generation of an r-planar BDD.

(Proof) By Lemma 3.1, there is a planar BDD for ψ_i (a complete symmetric decision diagram). By Lemma 3.5, there is an r-planar BDD for S_k (a voting function). As shown in Fig. 3.7, consider the BDD where the upper block realizes ψ_i's, and the lower block realizes S_k's. By connecting appropriate terminals between two blocks, we have an r-planar BDD for the function f. (Q.E.D.)

Example 3.3 *Consider the function $f = (x_1 \oplus x_2)x_3x_4 \vee x_1x_2(x_3 \vee x_4)$. f is partially symmetric with respect to $X_1 = (x_1, x_2)$ and $X_2 = (x_3, x_4)$. Note that f can be represented as $f(X_1, X_2) = \psi_0(X_1)S_3(X_2) \vee \psi_1(X_1)S_2(X_2) \vee \psi_2(X_1)S_1(X_2)$, where $\psi_0(X_1) = \bar{x}_1\bar{x}_2$, $\psi_1(X_1) = x_1 \oplus x_2$, $\psi_2(X_1) = x_1x_2$, $S_3(X_2) = 0$, $S_2(X_2) = x_3x_4$, and $S_1(X_2) = x_3 \vee x_4$. Thus, by Theorem 3.2, f has an r-planar BDD, as shown in Fig. 3.8.* *(End of Example)*

Corollary 3.1 *A monotone increasing threshold function having at most two different weights has an r-planar BDD.*

(Proof)

1) A monotone increasing threshold function f having only one weight is a voting function. Thus, by Lemma 3.2, f has an r-planar BDD.

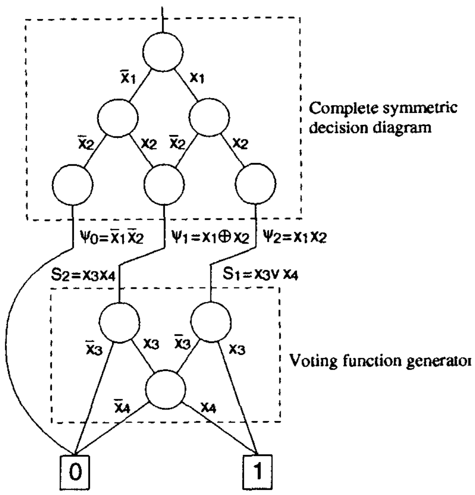


Figure 3.8: r-planar BDD for $f = (x_1 \oplus x_2)x_3x_4 \vee x_1x_2(x_3 \vee x_4)$.

2) Suppose that f has a characteristic vector $(w_1, w_2, \ldots, w_k, w_{k+1}, \ldots, w_n : T)$, where $w_1 = w_2 = \cdots = w_k$, and $w_{k+1} = w_{k+2} = \cdots = w_n$. In this case, f is partially symmetric with respect to $X_1 = (x_1, x_2, \ldots, x_k)$ and $X_2 = (x_{k+1}, x_{k+2}, \ldots, x_n)$, and f can be represented in the form (3.2). Since f is a monotone increasing function, we can assume that $S_{a_i}(X_2) \subseteq S_{a_{i+1}}(X_2)$. Thus, by Theorem 3.2, f has an r-planar ROBDD. (Q.E.D.)

Lemma 3.6 *Let $X = (x_1, x_2, \ldots, x_n)$. Let $\phi_i(X)(i = 0, 1, \ldots, t)$ be threshold functions with a characteristic vector $(w_1, w_2, \ldots, w_n : T)$, where $w_n = 1$ and*

$$w_i \geq \sum_{r=i+1}^{n} w_r, \text{ and } \phi_i(X) \supseteq \phi_{i+1}(X).$$

Then, both $\psi_i(X) = \phi_i(X) \cdot \overline{\phi_{i+1}(X)}$ $(i = 1, 2, \ldots, t-1)$ and $\psi_t(X) = \phi_t(X)$ can be represented in an r-planar BDD.

Example 3.4 *Consider the threshold functions with characteristic vector (2,1,1:T). In this case,*

$$\begin{aligned}
\phi_0(X) &= 1 & (T = 0)\\
\phi_1(X) &= x_1 \vee x_2 \vee x_3 & (T = 1)\\
\phi_2(X) &= x_1 \vee x_2x_3 & (T = 2)\\
\phi_3(X) &= x_1(x_2 \vee x_3) & (T = 3)\\
\phi_4(X) &= x_1x_2x_3 & (T = 4).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi_4(X) &= \phi_4(X) = x_1x_2x_3\\
\psi_3(X) &= \phi_3(X) \cdot \overline{\phi_4(X)} = x_1(x_2 \oplus x_3)\\
\psi_2(X) &= \phi_2(X) \cdot \overline{\phi_3(X)} = \bar{x}_1x_2x_3 \vee x_1\bar{x}_2\bar{x}_3\\
\psi_1(X) &= \phi_1(X) \cdot \overline{\phi_2(X)} = \bar{x}_1(x_2 \oplus x_3)\\
\psi_0(X) &= \phi_0(X) \cdot \overline{\phi_1(X)} = \bar{x}_1\bar{x}_2\bar{x}_3.
\end{aligned}$$

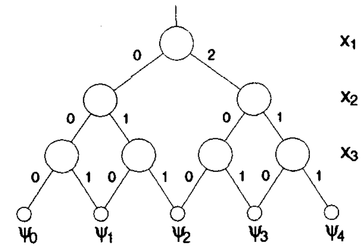


Figure 3.9: BDD generating ψ_i's.

Fig. 2.4(a) shows the complete decision tree. By merging the terminal nodes that represent the same weight-sum, we have a BDD as shown in Fig. 3.9. *(End of Example)*

Theorem 3.3 *Suppose that $X = (X_1, X_2)$ is a partition of variables $X = (x_1, x_2, \ldots, x_n)$. If a function f can be represented as*

$$f(X_1, X_2) = \bigvee_{i=0}^{t} \psi_i(X_1)S_{a_i}(X_2), \tag{3.3}$$

where $S_{a_i}(X_2)$ is a symmetric function satisfying $S_{a_i}(X_2) \subseteq S_{a_{i+1}}(X_2)$, and ψ_i $(i = 1, 2, \ldots, t)$ is function as defined in Lemma 3.6, then f has an r-planar BDD.

(Proof) We can prove this theorem in a similar way to Theorem 3.2. (Q.E.D.)

Corollary 3.2 *Suppose that a monotone increasing threshold function f has a characteristic vector $(w_1, w_2, \ldots, w_k, w_{k+1}, \ldots, w_n : T)$, where $w_k = 1$,*

$$w_i \geq \sum_{j=i+1}^{k} w_j, \ (i = 1, 2, \ldots, k-1), \text{ and}$$

$w_k = w_{k+1} = \cdots = w_n$. Then, f has an r-planar ROBDD.

(Proof) Note that f can be written in the form (3.3). Because f is monotone increasing, we can assume that $S_{a_i}(X_2) \subseteq S_{a_{i+1}}(X_2)$. Thus, by Theorem 3.3, f has an r-planar ROBDD. (Q.E.D.)

Example 3.5 *Consider the 5-variable function with the characteristic vector $(4, 3, 3, 2, 1 : 6)$. f is symmetric with respect to $X_2 = (x_2, x_3)$. Also, the weights for $X_1 = (x_1, x_4, x_5)$ satisfy the conditions of Lemma 3.6. Thus, f can be represented as*

$$f = \bigvee_{i=0}^{7} \psi_i(X_1)S_{a_i}(X_2).$$

Fig. 3.10 shows the r-planar BDD for f. The upper block generates ψ_i, and the lower block generates S_{a_i}. Note that each edge has a weight. In each path from the root node to the constant 1, the sum of the weights is greater than or equal to 6. On the other hand, in each path from the root node to the constant 0, the sum of the weights is less than 6. We can reduce the BDD without introducing crossings. *(End of Example)*

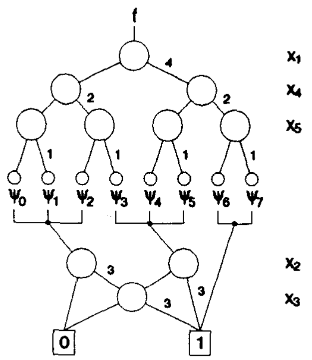


Figure 3.10: BDD for a threshold function.

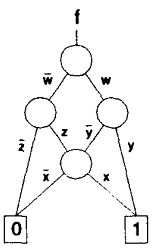


Figure 3.11: ROBDD for $f = w(x \vee y) \vee xz$.

Theorem 3.4 *All the monotone increasing functions up to four variables have r-planar ROBDDs.*

(Proof) From the table of NPN-representative functions of four variables [5], we can identify all the monotone increasing functions. By using Theorem 2.2, Corollaries 3.1 and 3.2, we can verify that all the representative functions have r-planar BDDs, except for $g = w(x \vee y) \vee xz$. Also, we can show that g has an r-planar BDD as shown in Fig. 3.11. (Q.E.D.)

Theorem 3.5 *All the monotone increasing threshold functions up to five variables have r-planar ROBDDs.*

(Proof) From the table of D-representative functions of NPN-equivalence classes up to five variables [9], we can verify the theorem. There are 62 representative functions. By using Theorem 2.2, Corollaries 3.1 and 3.2, we can show that 59 functions have r-planar BDDs. For the other 3 functions, we obtained their r-planar BDDs by inspection. (Q.E.D.)

4 Conclusion and Comments

In this paper, we presented the concept of r-planar MDDs and BDDs. Then, we showed classes of functions that have r-planar MDDs and BDDs.

Throughout this paper, we assumed that 1-edges emerge to the right and 0-edges emerge to the left. As a result, the realized functions are monotone increasing. By lifting this restriction, we can realize unate functions with r-planar BDDs. Specifically, given a unate function $f(X)$, we can convert it into a monotone increasing function by complementing variables. The converse operation of converting a monotone increasing function to a unate function, can be accomplished in the domain of BDDs, by interchanging 0 and 1 labels on all edges associated with some variable. This is the same as having 0-edges emerge to the right and 1-edges to the left. Thus, with minor modification, the results presented here can be made to apply to unate functions.

For a given monotone increasing function, in most cases, we can find an r-planar BDD among minimum BDDs (i.e., BDDs having the least number of nodes). However, some functions require additional nodes to make their BDDs r-planar. In the past, reduction of the number of nodes was the major subject in the optimization of BDDs. However, in implementing multilevel networks directly from the BDDs, the planarity of BDDs is also important, since crossing produces delay in LSIs. It is interesting to extend the theory for the decision diagrams with EXOR operators [13].

Acknowledgments

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